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Generalized Enveloping Algebras and Quantum Kinematic Coherent States of Noncompact Lie Groups

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A new concept of "generalized enveloping algebra" is introduced by means of the generalized Heisenberg commutation relations of non-Abelian quantum kinematics. This concept is examined within the quantum-kinematic formalism of some noncompact Lie groups of a special kind. The well known Gel'fand theorem (which relates the center of the traditional enveloping algebra with the adjoint representation) is then extended to the generalized enveloping algebra of the group. In this way, the isomorphism of the "generalized left-center" and the "traditional right-center" of the corresponding enveloping algebras is proved within the left regular representation of noncompact Lie groups of the chosen kind. As an interesting application of generalized enveloping algebras, this paper contains a brief discussion of quantum-kinematic (boson) ladder operators for non-Abelian noncompact finite Lie groups and of their corresponding coherent states.

1. INTRODUCTION

In a previous paper (Krause, 1991, hereafter referred to as paper I) the formalism of non-Abelian group quantization was briefly revisited within the regular representation of noncompact Lie groups (Krause, 1985). It was shown that such r-dimensional groups always have a set of r basic *quantum-kinematic invariant operators*, which substantially differ from the traditional invariants of the Lie algebra. The relation of the traditional invariants with the new quantum-kinematic invariants was also examined in paper I.

Hitherto all invariant operators of Lie group theory have been defined as *functions of the generators that commute with all the generators* of a given representation. For the sake of brevity we refer to this current notion as the *traditional invariants* of Lie group theory. In the present line of research, we

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are interested in studying a new type of invariant operators (i.e., *quantum-kinematic invariants*), which are defined as *functions of the generators and of the generalized position operators* (see below) *that commute with all the generators.*

In this paper we would like to study this subject further. In particular, here we shall discuss the *quantum-kinematic generalization* of a well known theorem (Gel'fand, 1950) concerning the center of the enveloping algebra and its relation to the adjoint representation of the corresponding Lie group (Barut and Raczka, 1977). For the motivation of the present work (in particular, for the huge physical possibilities of non-Abelian quantum kinematics) we refer the reader to paper I and to the literature quoted therein. Let us here only remark that the results in the traditional theory of invariants are both remarkably successful and faintly distressing because some Lie groups have no Casimir operator; other Lie groups have only transcendental invariant operators that do not belong to the enveloping algebra. Moreover, it also happens that some Lie groups have no traditional invariants at all.

As shown in paper I, one arrives at completely different results if one uses the "group quantization method" (Krause, 1985), for then it turns out that *every* r-dimensional Lie group has a set of r *basic quantum-kinematic operators, which commute with all the generators* of the group. Moreover, we have proven that once a Lie group has been "quantized" its basic kinematic invariant operators arise in a rather natural manner (even in those extreme cases where the group has *no* traditional invariant at all). In paper I we prove this fact for a special kind of noncompact Lie groups. Though this feature is valid also for other kinds of Lie groups, quantum kinematics of Lie groups, in general, sets a rather difficult issue. (We postpone the consideration of the general formalism of quantum-kinematic invariant operators to forthcoming papers.)

The importance of invariant operators in physics has long been known because, for the symmetry group of a given system, they yield the conservation laws and superselection rules obeyed by the system. In fact, superselection rules and characterization of the states of a system by means of the eigenvalues and eigenfunctions of invariant operators were emphasized by Wigner (1939) in his study of the traditional invariants for the Poincaré group. The same task was also performed by Levy-Leblond (1972) for the traditional invariants of the Galilei group. Traditional invariants have also been found (and applied in physics) for several groups containing the Poincaré group (Roman *et al.*, 1972), for some containing the Galilei group (Abellanos and Alonso, 1975), for the similitude groups of Minkowski space in four and three dimensions, as well as for the $O(4, 1)$ de Sitter group (Patera *et al.*, 1975, 1976), and probably for a few more nonsemisimple Lie groups. Moreover, it has long been well known that traditional invariant operators of dynamical groups (Bohm, 1991) yield mass formulas (Gell-Mann, 1962; Okubo, 1962), energy spectra (Bargmann, 1936), and other physical characterizations of several physical systems.

As we have already emphasized, *all* these interesting and successful applications of invariant operators have been rooted hitherto on Casimir operators, belonging to the traditional enveloping algebra attached to the corresponding Lie group (Casimir, 1931). As we shall see in this paper, for a given noncompact Lie group, this traditional set is *not* an *exhaustive set of invariant operators*. On the other hand, it will be shown that the *generalized enveloping algebra of quantum kinematics* (see below) does *always* yield an exhaustive set of invariant operators which are regular functions of the generators.

Furthermore, as a miscellaneous instance of the general utility of non-Abelian quantum kinematics (Krause, 1985), in this paper we do also apply this new formalism to build a complete set of r commuting boson "annihilation" and "creation" ladder operators, which belong to the generalized enveloping algebra of any given r-dimensional non-Abelian noncompact Lie group. The physical interest of the associated *generalized coherent states* (which will be exhibited below) is well understood (Perelomov, 1986).

Although we do not include physical applications in the space allotted in this paper, we deem these two features of the present approach to "group quantization" as sufficiently interesting to arouse the attention of physicists. Several applications will be published elsewhere. Here we shall dwell only on the theoretical formalism of these two issues for the case of noncompact Lie groups.

The organization of this paper is as follows. Section 2 contains a rather sketchy review of group quantization and includes some features on the basic quantum-kinematic invariants of noncompact Lie groups. In Section 3, we present the Gel'fand theorem, which states that the necessary and sufficient condition for an element of the traditional enveloping algebra to belong to the center of the Lie algebra is that the coefficients of the expansion are invariant tensors for the adjoint representation of the group. We easily prove this theorem within the present formalism. Next, in Section 4, we extend these notions, introducing the concept of the "generalized enveloping algebra" of the generators of the (left) regular representation. In this manner, the "generalized center" of the algebra affords an interesting generalization of the Gel'fand theorem. In Section 5, as a miscellaneous instance, we apply this formalism to build the complete set of r commuting quantum-kinematic boson ladder operators (see also Krause, 1992), and we find their associated generalized coherent states in Section 6. Finally, in

Section 7 we add some concluding remarks. The paper ends with a short Appendix concerning a matter of consistency.

2. NON-ABELIAN QUANTUM KINEMATICS REVISITED

We here repeat some of the main concepts leading to group quantization and non-Abelian quantum kinematics of noncompact Lie groups, because this new formalism is not known to most physicists (Krause, 1985). It is our intention to describe (without proof) only those features which are relevant for the discussion of the quantum-kinematic generalized enveloping algebras. In particular, here we shall *not* dwell on the possible physical meaning of the formalism (Krause, 1986, 1988).

Henceforth, G denotes a noncompact, connected and simply connected, r-dimensional non-Abelian Lie group (such as, for instance, the universal covering group of a noncompact Lie group). Furthermore, we shall *assume* that there exists a coordinate patch $q=(q^1,\ldots,q^r)$ which covers the *whole* group manifold $M(G)$ and maintains everywhere a one-to-one correspondence with the elements of G; i.e., the coordinates q^a, $a=1,\ldots,r$, are real and provide a set of r essential parameters of G. This is a strong condition, to be sure. However, as a matter of fact, most Lie groups of physical interest are of the type known as "linear Lie group," in the sense that they have at least one faithful finite-dimensional representation. It is well known that the *whole* of a connected linear Lie group of dimension r can be parametrized by r real numbers $q^1,\ldots,q^r$, which form a connected set in R^r. Of course, there is *no* requirement in general that this *global parametrization* of G be faithful. Nevertheless, there are many instances of noncompact, connected and simply connected linear Lie groups (of physical relevance) for which the global parametrization provides a one-to-one faithful mapping. For the sake of simplicity, and in order to concentrate on the issue of generalized quantum-kinematic enveloping algebras, in this paper we shall deal exclusively with Lie groups which satisfy this condition.

In the sequel we write $\bar{q}=\bar{q}(q)$ to denote that point in $M(G)$ which labels the inverse element corresponding to q, and $e=(e^1,\ldots,e^r)\in M(G)$ to denote the labels of the identity element. Of course, $M(G)$ carries an analytic mapping, $g\colon M(G)\times M(G)\to M(G)$, that is endowed with the group property of G. Hence, in this parametrization of G one has a well-defined set of r group-multiplication functions, $g^a(q';q)=q''^a\in M(G)$, which realize the group law in $M(G)$ (see, e.g., Racah, 1965).

Now, in order to *quantize the group* G let us associate the essential parameters q^a with a set of r commuting Hermitian operators Q^a, which act within the carrier space of the regular representation and may be interpreted as generalized "position" operators of the group manifold.

Thus, within the common rigged Hilbert space $\mathscr{H}(G)$ that carries both (left and right) regular representations, we next define the following spectral integrals over the group manifold (Krause, 1985, 1991):

$$Q^a = \int d^r q \, |q\rangle \, q^a \, \langle q| = \int d\mu_L(q) \, |q\rangle_L \, q^a \, \langle q|_L$$

$$= \int d\mu_R(q) \, |q\rangle_R \, q^a \, \langle q|_R \tag{2.1}$$

i.e., we set $Q^a = Q^a_L = Q^a_R$. The Q's are *generalized position operators* of $M(G)$, acting in $\mathscr{H}(G)$; in fact, one has

$$Q^a \, |q\rangle = q^a \, |q\rangle, \qquad Q^a \, |q\rangle_L = q^a \, |q\rangle_L, \qquad Q^a \, |q\rangle_R = q^a \, |q\rangle_R \tag{2.2}$$

Hence, the Q's provide a complete set of commuting Hermitian operators in $\mathscr{H}(G)$. Here we have used the *Hurwitz invariant measures* on $M(G)$; i.e., $d\mu_L(q) \equiv \mu_0 \bar{L}(q) \, d^r q$, $d\mu_R(q) \equiv \mu_0 \bar{R}(q) \, d^r q$, and $d^r q = dq^1 \cdots dq^r$. (In order to simplify the notation, in this paper we assume $\mu_L = \mu_R = \mu_0$, but this choice is not strictly necessary.) (See the Appendix of paper I for details, and for a unified formalism of the two regular representations which shall be assumed as theoretical framework in what follows).

In paper I we have shown that the set of r basic *quantum-kinematic invariant operators* correspond essentially to the generators of the right (left) regular representation acting as invariant operators within the left (right) regular representation of G. As was shown in that paper, this feature is possible if one "quantizes" the group (i.e., $q^a \to Q^a$), because only in this fashion do the basic quantum-kinematic invariant operators appear as linear combinations of the generators, whose matrix coefficients are *functions* of the *generalized position operators* Q^a of G. Indeed, it was found that in the left regular representation (for instance) the invariant operators are given by

$$R_a(Q; L) = R^\dagger_a(Q; L) = \bar{A}^b_a(Q) L_b - \tfrac{1}{2} i\hbar f^{\,b}_{ab} \tag{2.3}$$

where the L's are the generators, $\bar{A}^b_a(q) = A^b_a(\bar{q})$ are the entries of the matrix of the adjoint representation G_A of G, and $f^{\,c}_{ab}$ denotes the structure constants. In fact, from the (left) *generalized Heisenberg commutation relations* associated with G, namely (Krause, 1985)

$$[Q^a, Q^b] = 0 \tag{2.4}$$

$$[Q^a, L_b] = i\hbar R^a_b(Q) \tag{2.5}$$

$$[L_a, L_b] = -i\hbar f^{\,c}_{ab} L_c \tag{2.6}$$

it follows that

$$[R_a(Q; L), L_b] = 0, \qquad a, b = 1, \ldots, r \tag{2.7}$$

As is well known, one defines Lie's (right and left) vector fields as follows:

$$X_a(q) \equiv R_a^b(q)\,\partial_b, \qquad Y_a(q) \equiv L_a^b(q)\,\partial_b \tag{2.8}$$

where R_a^b and L_a^b are the (right and left) *transport matrices* for contravariant vectors in $M(G)$, which are obtained from $g^a(q'; q)$ in the usual "classical" fashion; i.e., $R_a^b(q) = \partial'_a\, g^b(q'; q)\,|_{q'=e}$, and $L_a^b(q) = \partial'_a\, g^b(q; q')\,|_{q'=e}$. In Eq. (2.5) we have defined

$$R_a^b(Q) = \int d\mu_L(q)\, |q\rangle_L\, R_a^b(q)\, \langle q|_L \tag{2.9}$$

and the L_a are the generators of the left regular representation; i.e.,

$$U_L(e + \delta q) = I - \left(\frac{i}{\hbar}\right) \delta q^a L_a \tag{2.10}$$

$$L_a\, |q\rangle_L = i\hbar X_a(q)\, |q\rangle_L \tag{2.11}$$

The Lie operators satisfy the Lie algebra:

$$[X_a(q), X_b(q)] = f^c_{\ ab} X_c(q), \qquad [Y_a(q), Y_b(q)] = -f^c_{\ ab}\, Y_c(q) \tag{2.12}$$

where the structure constants are given by $f^c_{\ ab} = R^c_{b,a}(e) - R^c_{a,b}(e) = L^c_{a,b}(e) - L^c_{b,a}(e)$.

In the present formalism we also need the *inverse transport matrices* in $M(G)$, which are defined by $\bar{R}_a^b(q) = \partial'_a(q'; \bar{q})\,|_{q'=q}$ and $\bar{L}_a^b(q) = \partial'_a\, g^b(\bar{q}; q')\,|_{q'=q}$. [Clearly, one has $R_a^b(e) = L_a^b(e) = \delta_a^b$ and $\bar{R}_a^c(q)\, R_c^b(q) = \bar{L}_a^c(q)\, L_c^b(q) = \delta_a^b$.] As a matter of fact, the following "mixed" transport matrices in $M(G)$ correspond to the *adjoint representation* of G (cf. Paper I):

$$A_a^b(q) = R_a^c(q)\, \bar{L}_c^b(q), \qquad \bar{A}_a^b(q) = L_a^c(q)\, \bar{R}_c^b(q) \tag{2.13}$$

since one has

$$A_a^c(q')\, A_c^b(q) = A_a^b[g(q'; q)] \tag{2.14}$$

and

$$A_a^b(e + \delta q) = \delta_a^b + \delta q^c f^b_{\ ca} \tag{2.15}$$

For proofs and more details concerning these results, we refer the reader to paper I. The main features concerning our interest here are shown in equations (2.3) and (2.7). Similar results hold for the right regular representations. Henceforth, for the sake of concreteness, we shall work only within the *left* regular representations of G.

3. THE GEL'FAND THEOREM

First, let us recall the traditional *enveloping algebra* $\mathbf{E}_L$ of the Lie algebra. By definition, this is the set of all those operators which are functions of the L's of the form

$$E(L)=E^0+E^aL_a+E^{(ab)}L_aL_b+E^{(abc)}L_aL_bL_c+\cdots \tag{3.1}$$

where the coefficients E^0, E^a, $E^{(ab)}$, $E^{(abc)}, \ldots$ are arbitrary constant c-numbers. One can always consider totally symmetrized coefficients in equation (3.1), because of the Lie algebra obeyed by the generators. [For instance, one easily gets

$$E^{ab}L_aL_b=E^{(ab)}L_aL_b-\tfrac{1}{2}i\hbar f^{\,a}_{bc}E^{[bc]}L_a \tag{3.2}$$

and so on.] Therefore, we can use the "totally reduced" form (3.1) for these operators, without loss of generality. As one knows, the *center* $\mathbf{C}_L$ of the enveloping algebra $\mathbf{E}_L$ is the set of all operators $C(L)\in\mathbf{E}_L$ which commute with all the generators:

$$[C(L), L_a]=0, \qquad a=1,\ldots,r \tag{3.3}$$

In this way, the Gel'fand theorem states that the *necessary* and *sufficient* condition for an element $C(L)$ of the enveloping algebra $\mathbf{E}_L$ to belong to the center $\mathbf{C}_L$ is that the coefficients, say C^0, C^a, $C^{(ab)}$, $C^{(abc)},\ldots$, are *invariant tensors* for the adjoint representation G_A of G. We can easily prove this theorem in the present formalism since, as is well known, one has

$$U_L^\dagger(q)\,L_a U_L(q)=A_a^b(q)L_b \tag{3.4}$$

where the U's are the representative operators of G (within the left regular representation) and the $r\times r$ matrix $A_a^b(q)$ carries the adjoint representation of G. Hence, if $C(L)$ is given by

$$C(L)=C^0+C^aL_a+C^{(ab)}L_aL_b+\cdots \tag{3.5}$$

one gets

$$U_L^\dagger(q)\, C(L)\, U_L(q) = C^0 + C^b A_b^a(q) L_a + C^{(cd)} A_c^a(q)\, A_d^b(q) L_a L_b + \cdots \tag{3.6}$$

and therefore a sufficient condition for $C(L) \in \mathbf{C}_L$ is that the coefficients satisfy

$$A_b^a(q) C^b = C^a, \qquad A_c^a(q) A_d^b(q) C^{(cd)} = C^{(ab)}, \ldots \tag{3.7}$$

Note that, due to the total symmetrization of the coefficients, this condition is also necessary (Barut and Raczka, 1977).

4. GENERALIZED ENVELOPING ALGEBRAS

We are now in a position to extend these notions, introducing the concept of the *generalized enveloping algebra* of the generators of the left regular representation of G. We shall denote this new enveloping algebra by $\mathbf{E}_L(Q)$. By definition, $\mathbf{E}_L(Q)$ contains all the operators $F(Q; L)$ which are functions of the Q's and the L's, and which are *regular* in the L's. So, let us write, for any element $F(Q; L) \in \mathbf{E}_L(Q)$,

$$F(Q; L) = F^0(Q) + F^a(Q) L_a + F^{(ab)}(Q) L_a L_b + \cdots \tag{4.1}$$

where now the coefficients are *operators* which one obtains from a corresponding set of c-number functions $F^0(q)$, $F^a(q)$, $F^{(ab)}(q), \ldots,$ defined on the group manifold $M(G)$. Here we take the "normal order" for the noncommuting Q's and L's, as shown in equation (4.1), because in this fashion, introducing the "Q-representation" of quantum kinematics (Krause, 1985), we can write

$$_L\langle q|\, F(Q; L)\, |\psi\rangle = F[q; -i\hbar X(q)]\, \psi_L(q) \tag{4.2}$$

where $\psi_L(q) = {}_L\langle q|\psi\rangle$, $|\psi\rangle \in \mathscr{H}(G)$, and where $F[q; -i\hbar X(q)]$ is the "scalar transformation" operator given by

$$F[q; -i\hbar X(q)] \equiv F^0(q) - i\hbar F^a(q)\, X_a(q) - \hbar^2 F^{(ab)}(q)\, X_a(q)\, X_b(q) + \cdots \tag{4.3}$$

It is advantageous to write the operators of $\mathbf{E}_L(Q)$ in the "reduced" form (4.1); that is, with totally symmetrized functions $F^{(ab)}$, $F^{(abc)}, \ldots,$ of the Q's, so that further reduction by means of the generalized commutation relations (2.5) is not possible.

In the same manner, we define the generalized enveloping algebra $\mathbf{E}_R(Q)$ of the generators R_a of the right regular representation (cf. paper I) as the set of operators of the "normal" form

$$F(Q;R)=F^0(Q)+F^a(Q)R_a+F^{(ab)}(Q)R_aR_b+\cdots \tag{4.4}$$

In this fashion, within the *left* quantum-kinematic formalism of G, one gets

$${}_L\langle q|\,F(Q;R)\,|\psi\rangle=F[q;\,-i\hbar Y(q)-\tfrac{1}{2}i\hbar(\mathrm{tr}\,f)]\,\psi_L(q) \tag{4.5}$$

where we have written $(\mathrm{tr}\,f)_a=f^{\,b}_{ab}$ [cf. paper I, equation (3.7)].

It is clear that there is a trivial one-to-one correspondence between the operators belonging in the two generalized enveloping algebras $\mathbf{E}_L(Q)$ and $\mathbf{E}_R(Q)$; namely, $F(Q;L)\leftrightarrow F(Q;R)$, where one uses the same function F. More interesting is the following (nontrivial) correspondence. According to equation (2.3), every operator $F(Q;R)\in\mathbf{E}_R(Q)$ can be written as an element $F_{\bar{A}}(Q;L)\in\mathbf{E}_L(Q)$; and *conversely*, every element $F(Q;L)\in\mathbf{E}_L(Q)$ can be written as $F_A(Q;R)\in\mathbf{E}_R(Q)$. Note that here we mean $F_{\bar{A}}(Q;L)\equiv F(Q;R)$ and $F_A(Q;R)\equiv F(Q;L)$, where the *forms* of the functions $F_{\bar{A}}$ and F_A are certainly *not* the same as that of F. Indeed, if $F(Q;R)\in\mathbf{E}_R(Q)$, using

$$[\bar{A}^c_a(Q),L_b]=-i\hbar f^{\,c}_{bd}\bar{A}^d_a(Q) \tag{4.6}$$

[which follows from equation (2.6) and the properties of the adjoint representation; see paper I, equation (2.17)], after some manipulations, one obtains

$$\begin{aligned}F(Q;R)&=[F^0(Q)-\tfrac{1}{2}i\hbar f^{\,b}_{ab}F^a(Q)-\tfrac{1}{4}\hbar^2 f^{\,c}_{ac}f^{\,d}_{bd}F^{(ab)}(Q)+\cdots]\\&\quad+[F^b(Q)-i\hbar f^{\,d}_{cd}F^{(bc)}(Q)+\cdots]\,\bar{A}^a_b(Q)L_a\\&\quad+[F^{(cd)}(Q)+\cdots]\,\bar{A}^a_c(Q)\,\bar{A}^b_d(Q)L_aL_b+\cdots\\&\equiv F^0_{\bar{A}}(Q)+F^a_{\bar{A}}(Q)L_a+F^{(ab)}_{\bar{A}}(Q)L_aL_b+\cdots\\&=F_{\bar{A}}(Q;L)\end{aligned} \tag{4.7}$$

which manifestly belongs to $\mathbf{E}_L(Q)$. (One proves the converse in a similar fashion.)

However, recalling the law of transformation of the generalized position operators of the group, i.e.,

$$U_L^\dagger(q)\,Q^aU_L(q)=g^a(q;Q) \tag{4.8}$$

[cf. paper I, equation (2.9)], it is immediate that for the operators belonging to $\mathbf{E}_R(Q)$ it follows that

$$U_L^\dagger(q)\,F(Q;R)\,U_L(q)=F[g(q;Q);R] \tag{4.9}$$

quite generally. Therefore, we see that only those operators which belongs to the *traditional enveloping algebra* $\mathbf{E}_R$ of the right-generators R_a yield invariant operators in the left regular representation of G. Moreover, these operators are functions of the Q's and the L's, which are regular in the L's, and therefore they pertain to the generalized enveloping algebra $\mathbf{E}_L(Q)$. In fact, let

$$E(R)=E^0+E^aR_a+E^{(ab)}R_aR_b+\cdots \tag{4.10}$$

be an element of the (traditional) enveloping algebra $\mathbf{E}_R$; then, by means of equation (4.7), we obtain the following *left-invariant* operator:

$$\begin{aligned}E(R)&=(E^0-\tfrac{1}{2}i\hbar f^b_{ab}E^a-\tfrac{1}{4}f^c_{ac}f^d_{bd}E^{(ab)}+\cdots)\\&\quad+(E^b-i\hbar f^d_{cd}E^{cb}+\cdots)\,\bar{A}^a_b(Q)L_a\\&\quad+(E^{(cd)}+\cdots)\,\bar{A}^a_c(Q)\,\bar{A}^b_d(Q)L_aL_b+\cdots\\&\equiv E^0_{\bar{A}}+E^b_{\bar{A}}\bar{A}^a_b(Q)L_a+E^{(cd)}_{\bar{A}}\bar{A}^a_c(Q)\,\bar{A}^b_d(Q)L_aL_b+\cdots\\&=E_{\bar{A}}(Q;L)\end{aligned} \tag{4.11}$$

Note that, contrary to the traditional contents of Gel'fand's theorem, this is an invariant operator for the left regular representation even if the coefficients E^a, $E^{(ab)},\ldots,$ in equation (4.10) are *not invariant tensors* for G_A; in other words, even if $E(R)\notin\mathbf{C}_R$.

Conversely, if one considers an operator of $\mathbf{E}_L(Q)$ of the special form

$$E(Q;L)=E^0+E^b\bar{A}^a_b(Q)L_a+E^{(cd)}\bar{A}^a_c(Q)\,\bar{A}^b_d(Q)L_aL_b+\cdots \tag{4.12}$$

where the E's are arbitrary c-numbers [which is an invariant operator of the left regular representation, as can be proved by means of the group property of the adjoint matrix $A^b_a(q)$], then it is easy to show that such an operator belongs to the traditional enveloping algebra $\mathbf{E}_R$. Furthermore, it can be proved also (though the proof is not so simple) that all the elements $E(Q;L)$ of the generalized enveloping algebra $\mathbf{E}_L(Q)$ which satisfy the invariance law

$$U_L^\dagger(q)\,E(Q;L)\,U_L(q)=E(Q;L) \tag{4.13}$$

are necessarily of the form stated in equation (4.12).

Hence, we have shown that all operators of the form defined in equation (4.12) constitute the *generalized center* $\mathbf{C}_L(Q)$ [of the generalized enveloping algebra $\mathbf{E}_L(Q)$], since they commute with all the generators; i.e.,

$$[E(Q; L), L_a] = 0, \qquad a = 1, \ldots, r \tag{4.14}$$

Moreover, we have shown also that there is an isomorphism $\mathbf{C}_L(Q) \sim \mathbf{C}_R$ between the generalized "left-center" and the traditional "right-center" of the corresponding enveloping algebras. [Clearly, one proves the isomorphism $\mathbf{C}_R(Q) \sim \mathbf{C}_L$ in the same manner.] These results provide an interesting *generalization of the Gel'fand theorem*, since the coefficients E^0, E^a, $E^{(ab)}, \ldots$ in equation (4.12) are completely arbitrary c-numbers (namely, *they are not necessarily invariant tensors of the adjoint representation* G_A).

In particular, it is also interesting to observe that, according to the Gel'fand theorem, if $C(L)$ belongs to the traditional center $\mathbf{C}_L$, then from equations (3.7) and (4.12), it follows that $C(L) \in \mathbf{C}_R$; and moreover, one has

$$C(L) = C(R) \tag{4.15}$$

[because in this case one proves $C_A(R) = C_{\bar{A}}(L)$]. This fact is well known indeed (Chen, 1989). However, its importance for non-Abelian quantum kinematics should be stressed, since the *superselection rules* provided by the invariant operators belonging to the traditional center $\mathbf{C}_L = \mathbf{C}_R$ [of the generalized enveloped algebras $\mathbf{E}_L(Q)$ and $\mathbf{E}_R(Q)$] play an outstanding role in physical applications (Krause, 1986, 1988).

5. GENERALIZED QUANTUM-KINEMATIC BOSON LADDER OPERATORS

As an important application of the formalism of generalized quantum-kinematic enveloping algebras, let us consider the possibility of having a set of r first-order linear operators belonging to $\mathbf{E}_L(Q)$, of the general form

$$\hat{a}_a(Q; L) = A_a(Q) + iB_a^b(Q) L_b, \qquad \hat{a}_a^\dagger(Q; L) = A_a(Q) - iL_b B_a^b(Q) \tag{5.1}$$

endowed with the following fundamental commutation properties:

$$[\hat{a}_a, \hat{a}_b] = 0, \qquad [\hat{a}_a, \hat{a}_b^\dagger] = \delta_{ab} \tag{5.2}$$

for a, $b = 1, \ldots, r$. In another paper (Krause, 1992, paper II henceforth), we have proven that such a set of boson ladder operators exists, notwithstanding the fact that G is a *non-Abelian* Lie group (of the special *noncompact* kind introduced in Section 2). Furthermore, since $A_a(q)$ and $B_a^b(q)$ must be real regular functions of the q's everywhere on $M(G)$, we have also proven that these operators are unique (within the addition of arbitrary constant multiples of the identity). Moreover, one can calculate them explicitly for any given noncompact Lie group of the assumed kind. In this way, one gets a complete set of non-Hermitian ladder operators acting in the Hilbert space that carries the regular representation of G, whose eigenvectors can be found, quite generally, as a system of special functions defined in $M(G)$.

Taking into account the generalized Heisenberg commutation relations associated with a non-Abelian G [equations (2.4)–(2.6)], as well as the definitions of Lie vector fields acting on $M(G)$ [given in equation (2.8)], a straightforward calculation yields the following system of coupled nonlinear differential equations for the coefficients of the ladder operators (see paper II):

$$B_a^c(q)\,X_c(q)\,A_b(q)-B_b^c(q)\,X_c(q)\,A_a(q)=0 \tag{5.3}$$

$$B_a^d(q)\,X_d(q)\,B_b^c(q)-B_b^d(q)\,X_d(q)\,B_a^c(a)+f^c_{de}B_a^d(q)\,B_b^e(q)=0 \tag{5.4}$$

$$B_a^c(q)\,X_c(q)\,A_b(q)+B_b^c(q)\,X_c(q)\,A_a(q)-\hbar B_a^c(q)\,X_c(q)\,X_d(q)\,B_b^d(q)=\hbar^{-1}\delta_{ab} \tag{5.5}$$

for all $q\in M(G)$. These are necessary and sufficient conditions for the operators defined in equations (5.1) to be endowed with the desired fundamental commutation relations [equations (5.2)]. Of course, we are interested only in those solutions $A_a(q)$ and $B_a^b(q)$ that are regular everywhere on the group manifold, so that $|\langle\psi|\,\hat{a}_a\,|\psi\rangle|=|\langle\psi|\,\hat{a}_a^\dagger\,|\psi\rangle|$ remain finite for all $|\psi\rangle\in\mathscr{H}(G)$.

Now, in order to solve this problem, we use an indirect method. (The method we follow in the sequel is different from that used in paper II, and has the advantage of yielding the desired coherent states in a straightforward way.) To this end, let us recall the Abelian group of translations T_r in r dimensions, with the group law given by $q''^a=q'^a+q^a$, $a=1,\ldots,r$ [namely, we use *canonical parameters* $(-\infty<q^a<+\infty)$ for labeling the elements of T_r]. In this particular case, equations (5.3)–(5.5) become much simpler, and $A_a(q)=\alpha q^a+\gamma^a$, $B_a^b(q)=\beta\delta_a^b$, with $\alpha\beta=(2\hbar)^{-1}$ and $\gamma^a=$ arbitrary constants, yield a solution. So one has the *standard* set of boson ladder operators

$$\hat{a}_a(Q;P)=\frac{1}{\sqrt{2}}\left(Q^a+\frac{1}{\hbar}P_a\right),\qquad \hat{a}_a^\dagger(Q;P)=\frac{1}{\sqrt{2}}\left(Q^a-\frac{i}{\hbar}P_a\right) \tag{5.6}$$

as it must be. Here we have written P_a to denote the generators of T_r, i.e., $U_T(\delta q) = I - (i/\hbar)\,\delta q^a P_a$, with $[Q^a, P_b] = i\hbar\delta^a_b$, as usual. The fact that the standard ladder operators are the *only admissible solution* (within arbitrary γ's) to equations (5.3)–(5.5) when $G = T_r$ follows because the A's and the B's must be regular everywhere (in particular, at $q = 0$), and because consistency demands $\langle\psi|\,\hat{a}_a\,|\psi\rangle^* = \langle\psi|\,\hat{a}_a^\dagger\,|\psi\rangle$. In this sense the familiar ladder operators for T_r shown in equation (5.6) are essentially the *unique* solution of the problem when $G = T_r$. (We set $\gamma^a = 0$, for all $1 \leqslant a \leqslant r$, without loss of generality.)

We next solve our problem for a *non-Abelian noncompact* r-dimensional G, using therefore the canonical parameters q^a (i.e., Cartesian coordinates) in the noncompact, connected and simply connected, group manifold $M(G)$. With this aim we use the conceptual framework presented in the Appendix of paper I (we refer the reader to that article in order to have a good understanding of the details that follow). Since $P_a\,|q\rangle = i\hbar\partial_a\,|q\rangle$, using the deefinition $|q\rangle_L = [\mu_0^{-1} L(q)]^{1/2}\,|q\rangle$ [see paper I, equation (A.6)], we obtain the following realization of P_a when operating on the kets $|q\rangle_L$ of the *left* (rigged) continuous basis:

$$P_a\,|q\rangle_L = \{\bar{R}^b_a(q)\,L_b + \tfrac{1}{2} i\hbar[\ln \bar{L}(q)],\, a\}\,|q\rangle_L \tag{5.7}$$

Hence, within the left regular representation of G, we can define the following operators that belong to $\mathbf{E}_L(Q)$:

$$P_a = \bar{R}^b_a(Q)\,L_b - i\hbar\{X_b(Q)\,\bar{R}^b_a(Q) - \tfrac{1}{2}[\ln \bar{L}(Q)],\, a\} \tag{5.8}$$

from which, using $P_a^\dagger = P_a$, we get

$$X_b(q)\,\bar{R}^b_a(q) = [\ln \bar{L}(q)],\, a \tag{5.9}$$

and thus we obtain

$$P_a = \bar{R}^b_a(Q)\,L_b - \tfrac{1}{2} i\hbar X_b(Q)\,\bar{R}^b_a(Q) \tag{5.10}$$

[Conversely, the condition stated in equation (5.9) can be proved in a direct manner, and hence $P_a^\dagger = P_a$ follows.] In this fashion, one sees that the standard ladder operators associated with a non-Abelian noncompact Lie group G can be cast in the following forms, within the left regular representation:

$$\hat{a}_a(Q; L) = \frac{1}{\sqrt{2}}\left\{\left[Q^a + \frac{1}{2} X_b(Q)\,\bar{R}^b_a(Q)\right] + \frac{i}{\hbar}\,\bar{R}^b_a(Q)\,L_b\right\} \tag{5.11a}$$

$$\hat{a}_a^\dagger(Q; L) = \frac{1}{\sqrt{2}}\left\{\left[Q^a - \frac{1}{2} X_b(Q)\,\bar{R}^b_a(Q)\right] - \frac{i}{\hbar}\,\bar{R}^b_a(Q)\,L_b\right\} \tag{5.11b}$$

In other words, we have found that

$$A_a(q)=\frac{1}{\sqrt{2}}\left\{q^a+\frac{1}{2}X_b(q)\,\bar{R}^b_a(q)\right\},\qquad B^b_a(q)=\frac{1}{\sqrt{2}}\hbar^{-1}\bar{R}^b_a(q) \quad (5.12)$$

yield the desired solution to equations (5.3)–(5.5). [One can prove this result quite generally, without recourse to the group T_r, and without assuming "canonical" parameters (see paper II).] Note that, for any given $|\psi\rangle\in\mathscr{H}(G)$, one has

$$\begin{aligned}\langle\psi|\,\hat{a}_a\,|\psi\rangle&=\frac{1}{\sqrt{2}}\int d^rq\,\psi^*(q)(q^a+\partial_a)\,\psi(q)\\&=\frac{1}{\sqrt{2}}\int d\mu_L(q)\,\psi_L^*(q)\left\{q^a+\frac{1}{2}[X_b(q)\,\bar{R}^b_a(q)]\right.\\&\qquad\left.+\bar{R}^b_a(q)\,X_b(q)\right\}\psi_L(q)\end{aligned} \quad (5.13a)$$

$$\begin{aligned}\langle\psi|\,\hat{a}_a^\dagger\,|\psi\rangle&=\frac{1}{\sqrt{2}}\int d^rq\,\psi^*(q)(q^a+\partial_a)\,\psi(q)\\&=\frac{1}{\sqrt{2}}\int d\mu_L(q)\,\psi_L^*(q)\left\{q^a-\frac{1}{2}[X_b(q)\,\bar{R}^b_a(q)]\right.\\&\qquad\left.-\bar{R}^b_a(q)\,X_b(q)\right\}\psi_L(q)\end{aligned} \quad (5.13b)$$

where one defines $\psi(q)=\langle q|\psi\rangle$ and $\psi_L(q)={}_L\langle q|\psi\rangle$; i.e., $\psi_L(q)=[\mu_0^{-1}L(q)^{1/2}\,\psi(q)$. (see the Appendix in paper I for these details; see also the Appendix of the present paper).

6. QUANTUM-KINEMATIC COHERENT STATES

In this fashion, one is ready to find the simultaneous eigenvectors $|\mathbf{n}\rangle=|n_1,\ldots,n_r\rangle$ such that

$$\hat{a}_a^\dagger\hat{a}_a\,|\mathbf{n}\rangle=n_a\,|\mathbf{n}\rangle \quad (6.1)$$

holds for $a=1,\ldots,r$, within the left regular representation of G. (We did not consider this subject in paper II. Here we shall fill this gap.) One defines $\langle q\,|\,\mathbf{n}\rangle=\phi_n(q)=\phi_{n_1}(q^1),\ldots,\phi_{n_r}(q^r)$, where each $\phi_{n_a}(q^a)$ denotes the corresponding ordinary one-dimensional Hermite orthogonal function, for $n_a=0,1,2,\ldots,$ with $a=1,\ldots,r$. Thus, we write

$$|\mathbf{n}\rangle=\int d^rq\,\phi_{\mathbf{n}}(q)\,|p\rangle=\int d\mu_L(q)\,\phi_{\mathbf{n}}^{(L)}(q)\,|q\rangle_L \quad (6.2)$$

where the *generalized r-dimensional Hermite functions* of G are just given by

$$\phi_{\mathbf{n}}^{(L)}(q) = {}_L\langle q \,|\, \mathbf{n} \rangle = [\mu_0^{-1} L(q)]^{1/2}\, \phi_{\mathbf{n}}(q) \tag{6.3}$$

The consistency of these manipulations can be checked very easily by means of the following realization of the "creation" operators:

$$a_a^{\dagger}\, |q\rangle_L = \frac{1}{\sqrt{2}} \left\{ q^a + \frac{1}{2}\, \partial_a [\ln \bar{L}(q)] + \partial_a \right\} |q\rangle_L \tag{6.4}$$

Furthermore, since the orthogonal basis $\{|\mathbf{n}\rangle = |n_1, \ldots, n_r\rangle\}$ is complete in $\mathscr{H}(G)$, one has

$$U_L(q)\, |\mathbf{n}\rangle = \sum_{\mathbf{m}} \Lambda_{\mathbf{m}\cdot\mathbf{n}}(q)\, |\mathbf{m}\rangle = \int d\mu_L(q')\, \phi_{\mathbf{n}}^{(L)}[g(\bar{q}; q')]\, |q'\rangle_L \tag{6.5}$$

where the group matrices

$$\begin{aligned} \Lambda_{\mathbf{m}\cdot\mathbf{n}}(q) &= \langle \mathbf{m}|\; U_L(q)\, |\mathbf{n}\rangle \\ &= \int d\mu_L(q')\, \phi_{\mathbf{m}}^{(L)}(q')\, \phi_{\mathbf{n}}^{(L)}[g(\bar{q}; q')] \\ &= \int d\mu_L(q')\, \phi_{\mathbf{m}}^{(L)}[g(q; q')]\, \phi_{\mathbf{n}}^{(L)}(q') \end{aligned} \tag{6.6}$$

are unitary and afford an infinite-dimensional reducible (left) representation of G. In fact, written more explicitly in terms of the Hermite functions, these matrices read

$$\Lambda_{\mathbf{m}\cdot\mathbf{n}}(q) = \mu_0^{-1} \int d\mu_{(L)}(q')\, \{L[g(q; q')]\, L(q')\}^{1/2}\, \phi_{\mathbf{m}}[g(q; q')]\, \phi_{\mathbf{n}}(q') \tag{6.7}$$

and therefore, using the completeness relation [paper I, equations (A.7)]

$$\sum_{\mathbf{n}} \phi_{\mathbf{n}}^{(L)}(q')\, \phi_{\mathbf{n}}^{(L)}(q) = {}_L\langle q' \,|\, q\rangle_L = \mu_0^{-1} L(q')\, \delta^{(r)}(q' - q) \tag{6.8}$$

we find that a rather lengthy (albeit straightforward) calculation yields

$$\sum_{\mathbf{p}} \Lambda_{\mathbf{m}\cdot\mathbf{p}}(q')\, \Lambda_{\mathbf{p}\cdot\mathbf{n}}(q) = \Lambda_{\mathbf{m}\cdot\mathbf{n}}[g(q'; q)] \tag{6.9}$$

as required. Interesting enough, written out in full detail, these matrices are given by Hurwitz G-invariant integrals of r-fold products of *ordinary*

Hermite orthogonal functions (suitably modulated) [cf. (6.7)]. Of course, in many applications, the evaluation of these integrals can be rather cumbersome, because of the presence of the factor

$$\phi_{\mathbf{m}}[g(q;q')]=\prod_{a=1}^{r} N_{m_a}H_{m_a}[g^a(q;q')]\exp\left\{-\tfrac{1}{2}\sum_{b=1}^{r}[g^b(q;q')]^2\right\} \quad (6.10)$$

in the integrands.

By the same token [that is, using $|q\rangle_L=[\mu_0^{-1}L(q)]^{1/2}\,|q\rangle$], one obtains immediately the *quantum-kinematic coherent states* associated with the left regular representation of G. One solve $\hat{a}_a\,|\mathbf{z}\rangle=2^{-1/2}z_a\,|\mathbf{z}\rangle$, where $|\mathbf{z}\rangle=|z_1,\ldots,z_r\rangle$; i.e., $\partial_a\langle q\,|\,\mathbf{z}\rangle=(z_a-q_a)\langle q\,|\,\mathbf{z}\rangle$. This yields the well-known standard coherent states of T_r in the Fock–Bargmann representation (Perelemov, 1986), namely

$$\psi(q;\mathbf{z})=\langle q\,|\,\mathbf{z}\rangle=c(\mathbf{z})\exp(-\tfrac{1}{2}\delta_{ab}q^aq^b+a_aq^a) \quad (6.11)$$

Hence, one also has $_L\langle q|\,\hat{a}_a\,|\mathbf{z}\rangle=2^{-1/2}z_a\,{}_L\langle q\,|\,\mathbf{z}\rangle$, from which it follows that

$$|\mathbf{z}\rangle=\frac{c(\mathbf{z})}{\sqrt{\mu_0}}\int d\mu_L(q)\,[L(q)]^{1/2}\exp\left(-\frac{1}{2}\delta_{ab}q^aq^b+a_aq^a\right)|q\rangle_L \quad (6.12)$$

which are the desired generalized coherent states for the non-Abelian noncompact G. Clearly, these most interesting kets deserve a more detailed study. In particular, their possible connections (if any) with Perelomov's generalized coherent states (Perelomov, 1986) will be discussed elsewhere.

By means of the same technique one can also define the generalized bosonic ladder operators which belong to the $\mathbf{E}_R(Q)$ enveloping algebra; that is, in terms of the quantum-kinematic invariant momenta $R_a(Q;L)$ (as operators within the left regular representation of G), instead of using the adjoint-vector momenta L_a as was done above (paper II). Furthermore, there is no difficulty in defining these generalized ladder operators within the central extension G by $U(1)$; i.e., for the regular *ray* representations of the group.

7. CONCLUDING REMARKS

As an important conclusion of this study, we see that the r basic quantum-kinematic invariant operators constitute an (almost) *exhaustive set of invariants* of G, in the sense that *every* invariant function $F(Q;L)$ which is *regular* in the L's is a function $f(R)$ that depends exclusively on the basic invariants $R_a(Q;L)$ of the (left) regular representation. As we have already shown in paper I, all the traditional invariant functions of the L's are of this kind.

Furthermore, the formalism admits the construction of operators $F(Q; L) \in \mathbf{E}_L(Q)$, which may be important both for mathematics and theoretical physics. For example, one may be interested in some functions $F_1(Q; L)$ and $F_2(Q; L)$ which satisfy some prescribed commutation relations, say

$$[F_1(Q; L), F_2(Q; L)] = F_{12}(Q; L) \tag{7.1}$$

where $F_{12}(Q; L)$ is a prescribed function. The non-Abelian quantum-kinematic analysis of (7.1) would give us the system of partial differential equations which one has to solve in order for this commutation relation to be possible at all. It is important to remark that this technique goes far beyond the canonical quantization procedure, which stems from changing a classical Poisson bracket into a commutator of quantum operators. The ladder operators of non-Abelian noncompact Lie groups presented in this paper are a good instance of this feature [equations (5.3)–(5.5)].

The physical applications of the generalized boson ladder operators of non-Abelian quantum kinematics settle an interesting question. As a matter of fact, in our previous work, quantum-kinematic ladder operators were already used rather successfully (Krause, 1986, 1988). It is indeed well known that several physically relevant Lie algebras can arise very naturally as bilinear products of boson annihilation and creation operators (see, e.g., Barut and Raczka, 1977). These Lie algebras are *intrinsic* to the quantized structure of the group G, and therefore they may yield interesting multiplets within the quantum-kinematic models (Krause, 1986, 1988), without recourse to direct or semidirect products in order to bring them to fore. [For a possible application of this feature in the quantized theory of the Poincaré group $P_+^\uparrow(1, 1)$, see Krause (1992).]

Hence, in view of the previous results, it seems that the generalized Heisenberg commutation relations of non-Abelian quantum kinematics [equation (2.5)] are group-theoretic tools worth further research both in physics and mathematics.

APPENDIX

Since much of what has been said here stems from the "intertwining formula" (see paper I)

$$|q\rangle_L = [\mu_0^{-1} L(q)]^{1/2} \, |q\rangle \tag{A.1}$$

one could question how well such a simple *change of scale* (in the continuous basis of the rigged Hilbert space) is able to produce so many far-reaching consequences. Indeed, a striking feature of this formula is that it produces, almost automatically, *easy solutions to otherwise very difficult problems.* Hence, the reader might have serious doubts about its consistency with the non-Abelian group structure of G. So it is our purpose here to provide a *consistency control* of the intertwining formula (A.1). This can be done very briefly. From the action of the (left) unitary operators of the group, namely

$$U_L(q')\,|q\rangle_L = |g(q';q)\rangle_L \tag{A.2}$$

and from Eq. (A.1) we get

$$U_L(q')\,|q\rangle = u(q';\hat{q})\,|g(q';q)\rangle \tag{A.3}$$

where we define

$$u(q';q) = \{\bar{L}(q)\,L[g(q';q)]\}^{1/2} \tag{A.4}$$

[Note that $\{|q\rangle\}$ is the continuous basis for the regular representation of T_r, in terms of canonical parameters; i.e., Cartesian coordinates in the assumed group flat-manifold.] Of course, equations (A.3) and (A.4) must be consistent with the (non-Abelian) group property of G; i.e.,

$$U_L(q')\,U_L(q) < U_L[g(q';q)] \tag{A.5}$$

This is indeed the case, since after some manipulations one obtains the following condition on $u(q';q)$:

$$u(q';q)\,u[q'';g(q';q)] = u[g(q'';q');q] \tag{A.6}$$

which is satisfied, according to the definition (A.4). This ends the required control.

By the way, this result shows neatly that *one and the same Hilbert space* that carries the regular representation of the Abelian group T_r (of rigid translations in the r-dimensional Cartesian scaffolding) also carries both regular representations of *all* those r-dimensional noncompact (connected and simply connected) Lie groups of the kind considered in this paper. It may also explain why the standard Heisenberg commutation relations

$$[Q^a, P_b] = i\hbar\delta^a_b \tag{A.7}$$

can be used as a valid rule of canonical quantization, even for those classical systems which do *not* manifest the translation symmetries described by

T_r, provided their symmetry properties can be englobed in a noncompact, connected and simply connected Lie group [such as, for instance, the simple harmonic oscillator; cf. Krause (1986)].

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